ON THE GENERAL EQUATIONS OF THE THEORY OF IDEAL PLASTICITY AND OF STATICS OF GRANULAR [PULVERULENT] MEDIA

(OB OBSHCHIKH URAVNENIIAKH TEORII IDEAL'NOI PLASTICHNOSTI I STATIKI SYPUCHEI SREDY)

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General equations for the theory of ideal plasticity were introduced by M. Levy [1]. Levy's representation of the Tresca-Saint Venant condition of plasticity by a single equation appeared to be quite cumbrous and was not subjected to a detailed investigation.

The Mises' condition of plasticity, when used for a general problem, makes it statically indeterminate and its solution involves considerable difficulties.

H. Hencky [2] solved certain problems and indicated that the application of the hypothesis of complete plasticity [3] makes axi-symmetrical problems statically determinate.

For an investigation of a space problem, W. Jenne [4] used the hypothesis of complete plasticity and the Mises' law of plastic flow. Jenne referred all his discussions to an isostatic coordinate system (system of principal stresses) and he obtained a set of relationships which is applicable to principal stresses and curvatures of isostatic curves. In doing this, Jenne ignored all the contradictions which would occur if the kinematic phase of the problem were to be considered.

A.U. Ishlinskii [5, 6] substantiated the hypothesis of complete plasticity by having shown that the relationships for complete plasticity are valid in the case when two out of three maximum shear stresses simultaneously reach their limiting value. In other words, an edge of Coulomb's prism corresponds to the state of complete plasticity. This prism interprets the Tresca-Saint Venant condition of plasticity in a space of the principal stresses σ_1, σ_2, σ_3. The same author [7] developed numerical methods for solution of axi-symmetrical problems. In a comparatively recent paper, R. Shield [8] provided a detailed analysis of an axi-symmetrical problem using

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Tresca-Saint Venant's conditions of plasticity. He solved a number of new problems and supplemented certain solutions in paper [7] by constructing velocity fields.

The present paper deals with development and analysis of the general equations of ideal plasticity. The Tresca-Saint Venant condition of plasticity and the associated law of plastic flow have been employed.

It is shown that a problem is statically determinate for cases when the plastic state of stress corresponds to an edge of Coulomb's prism.

The general equations of statics of granular media under conditions of complete limiting state are also considered in this paper. These are the cases when the limiting state of stress corresponds to an edge of the surface which interprets the condition of limiting equilibrium in the space of principal stresses. It is shown that under these conditions the general problem of the statics of granular media is statically determinate.

It should be noted that by the associated flow rule, a flow law should be understood, which is regarded as a plastic potential. In such a case the work performed by stresses on corresponding increments of plastic strains is a minimum and therefore such a development of the theory appears to be most correct and substantiated.

1. According to the Tresca-Saint Venant yield condition, plastic flow may appear when the maximum shear stress reaches a certain constant limiting value.

Obviously for the above condition of plasticity, only two types of plastic states of stress are possible; namely, the points σ_1, σ_2, σ_3 are located either on the edges of the prism or on its faces.

Then for any edge of the prism one of the conditions must be satisfied:

$$\sigma_i = \sigma_j = \sigma_k \pm 2k \tag{1.1}$$

and for its faces:

$$\begin{aligned} &\sigma_i = \sigma_j + 2k, \qquad && \sigma_i > \sigma_k > \sigma_j \\ &\sigma_j = \sigma_j - 2k, \qquad && \sigma_j < \sigma_k < \sigma_j \end{aligned} \tag{1.2}$$

The velocities of plastic strains for case (1.1) are determined from the condition of incompressibility

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$$

and from the condition of isotropy, which requires the coincidence of the principal axes of velocity tensors for strains and stresses.

From the accepted flow rule, it immediately follows for case (1.2) that $\epsilon_k = 0$. Therefore, the equation which determines velocities of plastic strains, assumes the form:

$$\varepsilon_i + \varepsilon_j = 0 \qquad (1.3)$$

Hence, for the second case, the velocity field for plastic strains becomes quite constrained. This leads to a certain generalized state of plane strain.

First consider relationship (1.1). With reference to the Cartesian system of coordinates x, y, z, let ξ, η, ζ be the directions of principal stresses σ_1, σ_2, σ_3.

The mutual orientation of these axes is defined by the direction cosines as indicated in the table

	ξ	η	ζ
x	l_1	m_1	n_1
y	l_2	m_2	n_2
z	l_3	m_3	n_3

Then, if

$$\sigma_1 = \sigma_2 = \sigma_3 \pm 2k \qquad (1.4)$$

then in derivations that follow, all the quantities having the dimensions of stress will be referred to a constant $\pm\ 2k$, thus obtaining

$$\sigma_x = \sigma_1 l_1^2 + \sigma_2 m_1^2 + \sigma_3 n_1^2, \ldots \qquad (1.5)$$
$$\tau_{xy} = \sigma_1 l_1 l_2 + \sigma_2 m_1 m_2 + \sigma_3 n_1 n_2, \ldots$$

Note that here and everywhere below, wherever it is convenient, the analogous expressions for the components: σ_y, σ_z, τ_{xz}, τ_{yz} etc. are not shown.

On the basis of (1.4), it follows from (1.5) that

$$\sigma_x = \sigma_1 + n_1^2, \qquad \sigma_y = \sigma_1 + n_2^2, \qquad \sigma_z = \sigma_1 + n_3^2$$
$$\tau_{xy} = n_1 n_2, \qquad \tau_{yz} = n_2 n_3, \qquad \tau_{zx} = n_3 n_1 \qquad (1.6)$$

One easily obtains:

$$\sigma_1 = \sigma - \tfrac{1}{3}, \qquad \sigma = \tfrac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$$

and, therefore, the three relationships among stresses may be obtained:

$$\tau_{xy}^2 = (\sigma_x - \sigma + \tfrac{1}{3})(\sigma_y - \sigma + \tfrac{1}{3})$$
$$\tau_{yz}^2 = (\sigma_y - \sigma + \tfrac{1}{3})(\sigma_z - \sigma + \tfrac{1}{3}) \qquad (1.7)$$
$$\tau_{zx}^2 = (\sigma_z - \sigma + \tfrac{1}{3})(\sigma_x - \sigma + \tfrac{1}{3})$$

or

$$\tau_{xy}\tau_{yz} = \tau_{zx}(\sigma_y - \sigma + \tfrac{1}{3})$$
$$\tau_{yz}\tau_{zx} = \tau_{xy}(\sigma_z - \sigma + \tfrac{1}{3})$$
$$\tau_{zx}\tau_{xy} = \tau_{yz}(\sigma_x - \sigma + \tfrac{1}{3}) \tag{1.8}$$

Also, obviously, there exist relationships

$$\tau_{xy}\tau_{yz}\tau_{zx} = (\sigma_x - \sigma + \tfrac{1}{3})(\sigma_y - \sigma + \tfrac{1}{3})(\sigma_z - \sigma + \tfrac{1}{3})$$
$$(\tau_{xy}\tau_{yz})^2 + (\tau_{yz}\tau_{zx})^2 + (\tau_{zx}\tau_{xy})^2 = \tau_{xy}\tau_{yz}\tau_{zx} \tag{1.9}$$

Assume $n_i = \cos\phi_i$ and substitute relationship (1.6) into the equations of equilibrium:

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} = 0, \ldots$$

to obtain:

$$\frac{\partial\sigma}{\partial x} - \sin 2\varphi_1 \frac{\partial\varphi_1}{\partial x} - \sin\varphi_1\cos\varphi_2\frac{\partial\varphi_1}{\partial y} - \tag{1.10}$$
$$- \sin\varphi_2\cos\varphi_1\frac{\partial\varphi_2}{\partial y} - \sin\varphi_1\cos\varphi_3\frac{\partial\varphi_1}{\partial z} - \sin\varphi_3\cos\varphi_1\frac{\partial\varphi_3}{\partial z} = 0, \ldots$$

whereby

$$\cos^2\varphi_1 + \cos^2\varphi_2 + \cos^2\varphi_3 = 1 \tag{1.11}$$

Equation (1.10), (1.11) for the characteristic surface of the system, when represented in the form $\psi(x, y, z)$, will provide

$$\Phi[2\Phi^2 - (\operatorname{grad}\psi)^2] = 0 \tag{1.12}$$

where

$$\Phi = \frac{\partial\psi}{\partial x}\cos\varphi_1 + \frac{\partial\psi}{\partial y}\cos\varphi_2 + \frac{\partial\psi}{\partial z}\cos\varphi_3$$

Since the vector grad ψ is perpendicular to the surface ψ, it follows from the equality $\Phi = 0$, that the direction of the vector ζ ($\cos\phi_1$, $\cos\phi_2$, $\cos\phi_3$) (which is the same as the direction of the principal stress σ_3)is a characteristic direction.

From the second relationship (1.12), it follows that

$$2(\operatorname{grad}\psi\cdot\zeta)^2 - (\operatorname{grad}\psi)^2 = 0$$

From this, it follows that the directions forming a 45° angle with the direction of σ_3 are the characteristic directions.

Hence, the system of equations (1.10), (1.11) will be always hyperbolic.

It is easily observed that the characteristic directions coincide with the planes of maximum shear stresses.

Consider again the condition of isotropy; according to this condition

$$\varepsilon_x = \varepsilon_1 l_1^2 + \varepsilon_2 m_1^2 + \varepsilon_3 n_1^2, \ldots, \qquad \varepsilon_{xy} = \varepsilon_1 l_1 l_2 + \varepsilon_2 m_1 m_2 + \varepsilon_3 n_1 n_2, \ldots \tag{1.13}$$

From the condition of incompressibility and from (1.13) for the case when $\epsilon_1 = \epsilon_2$ it follows that

$$\varepsilon_x = \varepsilon_1 (1 - 3n_1^2), \qquad \varepsilon_y = \varepsilon_1 (1 - 3n_2^2), \qquad \varepsilon_z = \varepsilon_1 (1 - 3n_3^2)$$
$$\varepsilon_{xy} = -3\varepsilon_1 n_1 n_2, \qquad \varepsilon_{yz} = -3\varepsilon_1 n_2 n_3, \qquad \varepsilon_{zx} = -3\varepsilon_1 n_3 n_1 \qquad (1.14)$$

From (1.14) and (1.6) it is easy to obtain

$$\frac{\varepsilon_x}{\sigma_x - \sigma} = \frac{\varepsilon_y}{\sigma_y - \sigma} = \frac{\varepsilon_z}{\sigma_z - \sigma} = \frac{\varepsilon_{xy}}{\tau_{xy}} = \frac{\varepsilon_{yz}}{\tau_{yz}} = \frac{\varepsilon_{zx}}{\tau_{zx}} \qquad (1.15)$$

Investigate further case (1.2). Let in the dimensionless variables

$$\sigma_1 - \sigma_2 = 1, \qquad \sigma_1 > \sigma_3 > \sigma_2 \qquad (1.16)$$

Introduce a certain curvilinear orthogonal system of coordinates α, β, γ. Let the mutual orientation of these axes and the axes ξ, η, ζ at each point be defined by the direction cosines. These cosines were given in the table above. Then

$$\sigma_\alpha = \sigma_1 l_1^2 + \sigma_2 m_1^2 + \sigma_3 n_1^2, \ldots, \qquad \tau_{\alpha\beta} = \sigma_1 l_1 l_2 + \sigma_2 m_1 m_2 + \sigma_3 n_1 n_2, \ldots \qquad (1.17)$$

Make use of (1.16) and (1.17) to obtain

$$\sigma_\alpha = \sigma_2 + l_1^2 + (\sigma_3 - \sigma_2) n_1^2, \ldots, \qquad \tau_{\alpha\beta} = l_1 l_2 + (\sigma_3 - \sigma_2) n_1 n_2 \qquad (1.18)$$

In an analogous way, from (1.3) and (1.13) obtain

$$\varepsilon_\alpha = \varepsilon_1 (l_1^2 - m_1^2), \ldots, \qquad \varepsilon_{\alpha\beta} = \varepsilon_1 (l_1 l_2 - m_1 m_2), \ldots \qquad (1.19)$$

Consider again the yield condition. Six relationships (1.18) and the three relationships between the direction cosines together contain eight variables σ_2, σ_3, l_i, n_i which can be eliminated. One can obtain expressions

$$q = -\tfrac{1}{3}(1 - c - c^2), \qquad r = -\tfrac{1}{27}(2 - 3c - 3c^2 + 2c^3)$$

where $c = \sigma_3 - \sigma_2$ and q and r are the second and the third invariants of stress deviator tensor, respectively.

$$q = S_\alpha S_\beta + S_\beta S_\gamma + S_\gamma S_\alpha - \tau_{\alpha\beta}^2 - \tau_{\beta\gamma}^2 - \tau_{\gamma\alpha}^2$$
$$r = S_\alpha \tau_{\beta\gamma}^2 + S_\beta \tau_{\gamma\alpha}^2 + S_\gamma \tau_{\alpha\beta}^2 - S_\alpha S_\beta S_\gamma - 2\tau_{\alpha\beta}\tau_{\beta\gamma}\tau_{\gamma\alpha}$$
$$S_\alpha = \sigma_\alpha - \sigma, \ldots \qquad \sigma = \tfrac{1}{3}(\sigma_\alpha + \sigma_\beta + \sigma_\gamma);$$

By eliminating the quantity c, obtain the desired M.Levy's yield condition

$$(4q + 1)(q + 1)^2 + 27r^2 = 0 \qquad (1.20)$$

By using condition (1.20) as a plastic potential, obtain

$$\varepsilon_\alpha = -\lambda \left[aS_\alpha - 54r\left(\tau_{\beta\gamma}^2 - S_\beta S_\gamma + \tfrac{1}{3} q\right)\right], \ldots \qquad (1.21)$$
$$\tau_{\alpha\beta} = -2\lambda \left[a\tau_{\alpha\beta} - 54r(\tau_{\alpha\beta} S_\gamma - \tau_{\beta\gamma}\tau_{\gamma\alpha})\right], \ldots$$

where

$$a = 6(q + 1)(2q + 1)$$

Consider the Tresca-Saint Venant yield condition as a limiting condition. Then, for the case of plane strain, it may be shown that

$$\sigma_3 = \frac{1}{2}(\sigma_1 + \sigma_2)$$

Hence, a case of plane strain is realized on the face of a prism along the line which is equidistant from the edges of this face.

However, the state of stress as close as desired to the plane strain state, may correspond to the edge of the prism.

Consider an infinitely long cylindrical body in the direction of the z-axis and subjected to a loading independent of z. Then this body will be in the state of plane strain. Orient the x and y normally to z and denote by u and v, the displacements along x- and y-axes, respectively. Let L be the body's contour in the xy-plane. Imagine a new body which would be formed by rotation of the contour L about some axis y_1 which is parallel to the y-axis. Denote by R the distance between these two axes. Assume that the loading in the plane on the contour L of the toroidal body coincides with the loading on the cylindrical body. Then for the toroidal body

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \varepsilon_z = \frac{u}{R + x}$$

Hence for any finite value of radius R, the component $\epsilon_3 = \epsilon_z \neq 0$ and the plastic state of stress corresponds to the edge of Coulomb's prism. However, for sufficiently large R the state of stress of the toroidal body approaches as closely as desired the state of stress for plane strain.

2. Consider general equations of statics for granular media under the condition of the complete limiting state.

The basic equation representing the state of granular media can be written in the form [9]

$$\max|\tau_n| = k + \sigma_n \operatorname{tg}\rho \tag{2.1}$$

where τ_n, σ_n are the tangential and normal stresses. k and ρ are constants.

It can be easily shown that condition (2.1) can be written in the form

$$|\sigma_i - \sigma_j| \leqslant 2k\cos\rho + (\sigma_i + \sigma_j)\sin\rho, \qquad i \neq j,\ i, j = 1, 2, 3 \tag{2.2}$$

In the space of principal stress σ_1, σ_2, σ_3 condition (2.2) can be interpreted by a hexagonal pyramid with a vertex at the point

$$\sigma_1 = \sigma_2 = \sigma_3 == -k \operatorname{ctg}\rho$$

All the lateral faces of this pyramid are inclined at the same angle with respect to the line $\sigma_1 = \sigma_2 = \sigma_3$.

It is obvious that any one of the lateral faces of this pyramid corresponds to the condition of complete limiting state. It is sufficient to consider any two diametrically opposite lateral faces.

Let $\sigma_1 = \sigma_2$, then for face A we obtain

$$\sigma_3 = \sigma_1\left(\frac{1+\sin\rho}{1-\sin\rho}\right) + \frac{2k\cos\rho}{1-\sin\rho} \tag{2.3}$$

and for face B

$$\sigma_3 = \sigma_1\left(\frac{1-\sin\rho}{1+\sin\rho}\right) - \frac{2k\cos\rho}{1+\cos\rho} \tag{2.4}$$

Having used relationships (1.5), we obtain for face A

$$\sigma_x = p + 2\left[\frac{p\sin\rho}{1-\sin\rho} + \frac{k\cos\rho}{1-\sin\rho}\right]n_1^2, \ldots$$

$$\tau_{xy} = 2\left[\frac{p\sin\rho}{1-\sin\rho} + \frac{k\cos\rho}{1-\sin\rho}\right]n_1 n_2; \ldots \tag{2.5}$$

where $p = \sigma_1$, and it is easy to arrive at

$$p = \frac{3\sigma(1-\sin\rho) - 2k\cos\rho}{3(1-\sin\rho) + 2\sin\rho}$$

The three relationships among stresses can be obtained as

$$\tau_{xy}^2 = (\sigma_x - p)(\sigma_y - p), \qquad \tau_{yz}^2 = (\sigma_y - p)(\sigma_z - p)$$
$$\tau_{zx}^2 = (\sigma_z - p)(\sigma_x - p)$$

or

$$\tau_{xy}\tau_{yz} = \tau_{zx}(\sigma_y - p), \qquad \tau_{yz}\tau_{zx} = \sigma_{xy}(\sigma_z - p)$$
$$\tau_{zx}\tau_{xy} = \tau_{yz}(\sigma_x - p)$$

Also

$$\tau_{xy}\tau_{yz}\tau_{zx} = (\sigma_x - p)(\sigma_y - p)(\sigma_z - p)$$
$$(\tau_{xy}\tau_{yz})^2 + (\tau_{yz}\tau_{zx})^2 + (\tau_{zx}\tau_{xy})^2 = 2\left[\frac{p\sin\rho + k\cos\rho}{1-\sin\rho}\right]\tau_{xy}\tau_{yz}\tau_{zx}$$

A system of four equations containing four unknowns p, ϕ_i may be obtained as follows: denote $n_i = \cos\phi_i$; substitute into equations of equilibrium the relationship (2.5) and introduce condition (1.11). The equation of the characteristic surface for this system of equations, when represented in the form $\psi(x,y,z)$, would provide

$$\Phi[a\Phi^2 - (\mathrm{grad}\,\psi)^2] = 0, \qquad a = \frac{2}{1-\sin\rho} \tag{2.6}$$

From (2.6) it follows that the directions, forming angles θ with the direction of the third principal stress are the characteristic directions, for which

$$\cos\theta = \pm\sqrt{{}^1/_2(1-\sin\rho)},$$

For face B we obtain

$$\cos\theta = \pm\sqrt{{}^1/_2(1+\sin\rho)}$$

Further, consider a general case when

$$\max\{|\tau_n| - f(\sigma_n)\} = 0 \tag{2.7}$$

Then the condition of the limiting state can be written in the form

$$\tfrac{1}{2}|\sigma_i - \sigma_j|\sin 2\omega \leqslant f\left[\tfrac{1}{2}(\sigma_i + \sigma_j) - \tfrac{1}{2}(\sigma_i + \sigma_j)\cos 2\omega\right] \quad (i \neq j,\ i, j = 1, 2, 3) \tag{2.8}$$

where $df/d\sigma_n = \cot 2\omega$.

It is obvious that in the space of the principal stresses σ_1, σ_2, σ_3, the condition of limiting equilibrium (2.8) may be interpreted by a certain curvilinear hexagonal pyramid, located symmetrically with respect to the line $\sigma_1 = \sigma_2 = \sigma_3$.

Consider two opposite faces of this pyramid and write the condition of the complete limiting state in the form

$$\sigma_1 = \sigma_2,\quad \tfrac{1}{2}(\sigma_1 - \sigma_3)\sin 2\omega \pm f\left[\tfrac{1}{2}(\sigma_1 + \sigma_3) - \tfrac{1}{2}(\sigma_1 - \sigma_3)\cos 2\omega\right] = 0$$

It is easy to obtain

$$\sigma_1 = \sigma_n + \frac{|\tau_n|(\cos 2\omega \pm 1)}{\sin 2\omega}, \qquad \sigma_3 = \sigma_n + \frac{|\tau_n|(\cos 2\omega \mp 1)}{\sin 2\omega} \tag{2.9}$$

From (2.3) and (1.5) obtain

$$\begin{aligned} \sigma_x &= \sigma_n + \frac{|\tau_n|}{\sin 2\omega}(\cos 2\omega \mp \cos 2\varphi_1) \\ \tau_{xy} &= \mp \frac{2|\tau_n|}{\sin 2\omega}(\cos\varphi_1 \cos\varphi_2) \end{aligned} \tag{2.10}$$

Inasmuch as $|\tau_n|$ and $|\omega|$may be expressed in terms of σ_n, the equations of equilibrium, relationships (2.10) and (1.11) lead to a system of four equations containing four unknowns σ_n and ϕ_i.

Following V.V. Sokolovskii [9], introduce a function S for which

$$dS = \frac{d\sigma_n}{|\tau_n|} - d\omega$$

Then it is easy to obtain

$$d\sigma_x = \frac{2|\tau_n|}{\sin^2 2\omega}(1 \mp \cos 2\omega \cos 2\varphi_1)\,dS \pm \frac{2|\tau_n|}{\sin 2\omega}\sin 2\varphi_1\, d\varphi_1$$

$$d\tau_{xy} = \mp 2\left\{\frac{2|\tau_n|}{\sin^2 2\omega}\cos 2\omega \cos\varphi_1 \cos\varphi_2 dS - \frac{|\tau_n|}{\sin 2\omega}\sin\varphi_1 \cos\varphi_2 d\varphi_1 - \frac{|\tau_n|}{\sin 2\omega}\cos\varphi_1 \sin\varphi_2 d\varphi_2\right\}$$

The differential equation of the characteristic surface $\psi(x, y, z)$ of the equations of equilibrium and relationship (1.11) may be represented in the form given by (2.6), where

$$a = \frac{2}{1 \pm \cos 2\omega}$$

Notice that for linear relationship (2.1)

$$\omega = \frac{\pi}{4} - \frac{\rho}{2}$$

To substantiate the anticipated attainment of a complete limiting state one should draw upon kinematics as is done in the theory of ideal plasticity. However, for the statics of granular media the above considerations have not been sufficiently developed as yet.

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